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Averages on the unitary group and applications to the problem of disordered conductors

Pier A Mello†

† Instituto de Física, Universidad Nacional Autonóma de México, Apartado Postal 20-364, Delegación A Obregón, 01000 Mexico, DF, Mexico

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Abstract. In a recent approach to the study of disordered conductors a maximum-entropy criterion is used to perform statistical calculations directly on the transfer matrix for the full physical system. Quantities of great importance in that approach are averages of products of the $\mathbf{u}$ and $\mathbf{u}^*$ matrix elements ($\mathbf{u}$ being a unitary matrix) evaluated with the invariant measure of the unitary group. In the present paper we evaluate those averages that contain up to four $\mathbf{u}$ and four $\mathbf{u}^*$ matrix elements, since they are the ones needed in the calculation of the average and covariance of transmission and reflection coefficients. The evaluation involves no direct integration at all, but makes use solely of the invariance of the measure, the fact that the matrix elements $u_{a\alpha}$ are commuting c-numbers and the unitarity of $\mathbf{u}$. The physical consequences for the average and covariance of transmission and reflection coefficients are briefly discussed.

1. Introduction

The usual approaches to the study of disordered conductors are based on a perturbation treatment or on numerical simulations. One calculates averages of *macroscopic* physical quantities (like the total conductance) in terms of the *microscopic* statistical distributions associated with the individual scatterers (the literature being very extensive, we only cite two review articles: Erdös and Herndon (1982) and Lee and Ramakrishnan (1985) (together with the references contained therein), and the recent paper by Lee *et al* (1987)). In various cases it was noted explicitly that different disordered microscopic Hamiltonians give the same final results (Lee and Stone 1985, Mello and Shapiro 1988). It is indeed the basic physical assumption behind the scaling approach to disordered conductors that the transport properties on a scale larger than the elastic mean free path l are insensitive to the microscopic origin of the disorder.

The above suggests the possibility of formulating a theory of disordered conductors which is independent of a particular choice for the disordered Hamiltonian. Such a theory was developed by Mello (1987, 1988), Mello *et al* (1988b), Mello and Stone (1990) (a closely related approach has been proposed by Imry (1986) and Muttalib *et al* (1987)) and is briefly reviewed below, in order to provide the physical motivation for the present paper.

Let the disordered system be placed between two perfect leads: there, the quantised transverse states define N channels for propagating modes, so that the wavefunction

† Also at: Departamento de Física, UAM-I, México, and Fellow of the Sistema Nacional de Investigadores, México.

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is specified by a $2N$-component vector. The transfer matrix† $\mathbf{M}$ relates the vector on the right with that on the left. It was shown by Mello *et al* (1988b), Mello and Stone (1990) and Mello and Pichard (1989), that the $\mathbf{M}$ matrices can be represented in the form

$$\mathbf{M}=\begin{bmatrix}\mathbf{u}^{(1)} & 0\\ 0 & \mathbf{u}^{(3)}\end{bmatrix}\begin{bmatrix}(1+\boldsymbol{\lambda})^{1/2} & \boldsymbol{\lambda}^{1/2}\\ \boldsymbol{\lambda}^{1/2} & (1+\boldsymbol{\lambda})^{1/2}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{(2)} & 0\\ 0 & \mathbf{u}^{(4)}\end{bmatrix}. \tag{1.1a}$$

Here, $\mathbf{u}^{(i)}$ ($i=1,\ldots,4$) are arbitrary $N\times N$ unitary matrices and $\boldsymbol{\lambda}$ is a real, diagonal matrix with N positive elements $\lambda_1,\ldots,\lambda_N$. If, in addition, the system is invariant under time reversal, we have the restrictions

$$\mathbf{u}^{(3)}=[\mathbf{u}^{(1)}]^* \qquad \mathbf{u}^{(4)}=[\mathbf{u}^{(2)}]^*. \tag{1.1b}$$

One can express the various quantities of interest in terms of these parameters. For instance, the $N\times N$ transmission and reflection matrices (when incidence is from the left) are given by

$$\mathbf{t}=\mathbf{u}^{(1)}\frac{1}{(1+\boldsymbol{\lambda})^{1/2}}\mathbf{u}^{(2)} \tag{1.2}$$

$$\mathbf{r}=-[\mathbf{u}^{(4)}]^{\dagger}\left(\frac{\boldsymbol{\lambda}}{1+\boldsymbol{\lambda}}\right)^{1/2}\mathbf{u}^{(2)}. \tag{1.3}$$

Transmission and reflection coefficients are then defined as

$$T_{ab}=|t_{ab}|^2 \qquad R_{ab}=|r_{ab}|^2 \tag{1.4}$$

t_{ab}, r_{ab} being the ab matrix elements of (1.2) and (1.3), respectively.

An ensemble of $\mathbf{M}$ matrices is described by the differential probability

$$\mathrm{d}P_L(\mathbf{M})=p_L(\mathbf{M})\,\mathrm{d}\mu(\mathbf{M}). \tag{1.5}$$

Here L is the length of the system and $\mathrm{d}\mu(\mathbf{M})$ the invariant measure of the group of $\mathbf{M}$ matrices, given by

$$\mathrm{d}\mu(\mathbf{M})=J_\beta(\boldsymbol{\lambda})\prod_c \mathrm{d}\boldsymbol{\lambda}_c\prod_{i=1}^{2\beta}\mathrm{d}\mu(\mathbf{u}^{(i)}) \tag{1.6a}$$

where

$$J_\beta(\boldsymbol{\lambda})=\prod_{a<b}|\lambda_a-\lambda_\beta|^\beta \tag{1.6b}$$

and $\beta=1,2$, depending on whether the system is or is not invariant under time reversal. In (1.6a), $\mathrm{d}\mu(\mathbf{u}^{(i)})$ is the invariant measure of the unitary group $\mathrm{U}(N)$, normalised so that $\int\mathrm{d}\mu(\mathbf{u}^{(i)})=1$.

The probability density $p_L(\mathbf{M})$ must satisfy an important combination law: if we put together two wires of lengths L and δL, with probability densities p_L and $p_{\delta L}$, the resulting probability density is given by the convolution $p_{L+\delta L}=p_L * p_{\delta L}$. The statistical distribution associated with systems of very small length δL is selected on the basis of a maximum-entropy criterion. The 'evolution' with length of the joint probability density $w_L(\boldsymbol{\lambda})$ of $\lambda_1,\ldots,\lambda_N$ is then governed by the Fokker–Planck or diffusion equation

$$l\frac{\partial w_L(\boldsymbol{\lambda})}{\partial L}=\frac{2}{\beta N+2-\beta}\sum_{a=1}^{N}\frac{\partial}{\partial\lambda_a}\left(\lambda_a(1+\lambda_a)J_\beta(\boldsymbol{\lambda})\frac{\partial}{\partial\lambda_a}\frac{w_L(\boldsymbol{\lambda})}{J_\beta(\boldsymbol{\lambda})}\right) \tag{1.7}$$

† The transfer matrix was designated by $\mathbf{R}$ in previous publications by the present author.

whereas the matrices $\mathbf{u}^{(i)}$ appearing in (1.1) are distributed according to the invariant measure $\mathrm{d}\mu(\mathbf{u}^{(i)})$ of the unitary group appearing in (1.6*a*). The $p_L(\mathbf{M})$ of (1.5) is thus independent of the $\mathbf{u}^{(i)}$ and will be said to be isotropic.

From the above model one can derive (Mello 1988, Mello *et al* 1988a, Mello and Stone 1990) the weak-localisation effect and the associated backscattering peak, the universal conductance fluctuations and the correlations in the transmission and reflection coefficients, in precise quantitative agreement with microscopic Green function calculations evaluated in the quasi-one-dimensional limit. Direct and crossed moments of transmission and reflection coefficients are thus our basic quantities, that we now examine in greater detail.

If the distribution of transfer matrices is isotropic, (1.2)-(1.7) imply that the moments mentioned in the previous paragraph can be expressed in terms of averages of functions of $\boldsymbol{\lambda}$ evaluated with the probability density $w_L(\boldsymbol{\lambda})$ of (1.7) and averages of products of $\mathbf{u}^{(i)}$, $\mathbf{u}^{(i)*}$ matrix elements evaluated with the invariant measure of the unitary group. We use the notation $\langle\ \rangle_L$ to indicate an average over $\boldsymbol{\lambda}$, while

$$\langle f(\mathbf{u})\rangle_0 = \int f(\mathbf{u})\,\mathrm{d}\mu(\mathbf{u}) \tag{1.8}$$

denotes an average on the unitary group, evaluated with the invariant measure $\mathrm{d}\mu(\mathbf{u})$. Using (1.8) we define the quantities†

$$Q^{a_1\alpha_1,\dots,a_l\alpha_l}_{b_1\beta_1,\dots,b_m\beta_m} = \langle (u_{b_1\beta_1}\dots u_{b_m\beta_m})(u_{a_1\alpha_1}\dots u_{a_l\alpha_l})^*\rangle_0 . \tag{1.9}$$

As an example we quote, in terms of this notation, the first and second moments of the transmission and reflection coefficients:

$$\langle T_{ab}\rangle_L^{(\beta)} = \sum_{\alpha\alpha'} Q^{a\alpha}_{a\alpha'} Q^{\alpha b}_{\alpha' b} \langle (\tau_\alpha \tau_{\alpha'})^{1/2}\rangle_L^{(\beta)} \tag{1.10}$$

$$\langle T_{ab}T_{a'b'}\rangle_L^{(\beta)} = \sum_{\alpha\gamma\alpha'\gamma'} Q^{a\alpha,a'\gamma}_{a\alpha',a'\gamma'} Q^{\alpha b,\gamma b'}_{\alpha' b,\gamma' b'} \langle (\tau_\alpha\tau_\gamma\tau_{\alpha'}\tau_{\gamma'})^{1/2}\rangle_L^{(\beta)} \tag{1.11}$$

$$\langle R_{ab}\rangle_L^{(\beta)} = \sum_{\alpha\gamma} \langle (\rho_\alpha\rho_\gamma)^{1/2}\rangle_L^{(\beta)} \times \begin{cases} Q^{\gamma a,\gamma b}_{\alpha a,\alpha b} & \beta=1 \\ Q^{\gamma a}_{\alpha a} Q^{\gamma b}_{\alpha b} & \beta=2 \end{cases} \tag{1.12}$$

$$\langle R_{ab}R_{a'b'}\rangle_L^{(\beta)} = \sum_{\alpha\gamma\alpha'\gamma'} \langle (\rho_\alpha\rho_\gamma\rho_{\alpha'}\rho_{\gamma'})^{1/2}\rangle_L^{(\beta)} \times \begin{cases} Q^{a\alpha,a'\gamma,b\alpha,b'\gamma}_{a\alpha',a'\gamma',b\alpha',b'\gamma'} & \beta=1 \\ Q^{a\alpha,a'\gamma}_{a\alpha',a'\gamma'} Q^{b\alpha,b'\gamma}_{b\alpha',b'\gamma'} & \beta=2 \end{cases} \tag{1.13}$$

where

$$\tau_\alpha = \frac{1}{1+\lambda_\alpha} \qquad \rho_\alpha = \frac{\lambda_\alpha}{1+\lambda_\alpha}. \tag{1.14}$$

The probability density $w_L(\boldsymbol{\lambda})$ needed to evaluate the $\boldsymbol{\lambda}$ averages in (1.10)-(1.13) is the solution of the diffusion equation (1.7), and the averages needed here were calculated approximately by Mello (1988) and Mello and Stone (1990). In contrast, the quantities Q of (1.9) are purely geometrical factors that can be evaluated exactly once and for all. *It is the purpose of the present paper* to evaluate the Q coefficients up to the order needed in (1.10)-(1.13). The results that we shall find here were quoted without proof by Mello *et al* (1988a) and Mello and Stone (1990), and were used to explicitly evaluate the average and covariance of transmission and reflection coefficients (see also Feng *et al* (1988) for a microscopic calculation of these quantities and Etemad *et al* (1986) for some experimental results).

† These quantities were called M by Mello *et al* (1988a).

Some of the Q coefficients of (1.9) corresponding to $l=m=1,2$ were calculated by Gaudin and Mello (1981) with an entirely different physical motivation: the statistical theory of nuclear reactions. Here we shall employ a more powerful method to evaluate *all* the Q coefficients corresponding to $l=m=1,2,3,4$.

The method that we shall use was devised by Mello and Seligman (1980) in connection with unitary and symmetric $\mathbf{S}$ matrices, again with a motivation coming from nuclear physics.

As we shall see, the evaluation of the integrals (1.9) will involve no 'direct' integration at all, but will make use solely of the invariance of the measure $d\mu(\mathbf{u})$, the fact that the matrix elements $u_{c\alpha}$ in (1.9) are commuting c-numbers, and the unitarity of the matrices $\mathbf{u}$: from these properties, three conditions (I, II, III, equations (2.3), (2.8), (2.9)) are derived in section 2 (an obvious consequence is, for instance, that $Q=0$ unless $l=m$). In section 3 it is found that, for $m=1,2$, those conditions give rise to linear equations with a unique solution, i.e. they define the problem uniquely.

In section 4 we indicate the most general form (proved in the appendix) that Q must take in order to satisfy condition I (directly associated with the invariance property of the measure); conditions II and III are then to be applied subsequently. That procedure is used in the same section to reproduce, as examples, the cases $m=1,2$ found in section 3, and to perform the calculation for $m=3$ and 4 in sections 5 and 6, respectively. In section 7 we summarise the physical implications for the problem of disordered conductors of the results obtained in the previous sections. Conclusions and a discussion of our results are given in section 8.

2. General properties of the Q coefficients

Let $\mathbf{u}^0$ be a fixed unitary matrix. Using, in (1.9), the transformation

$$\mathbf{u}=\mathbf{u}^0\mathbf{u}'=\mathbf{u}''\mathbf{u}^0 \tag{2.1}$$

together with the defining property

$$d\mu(\mathbf{u})=d\mu(\mathbf{u}')=d\mu(\mathbf{u}'') \tag{2.2}$$

of the invariant measure, we obtain the first condition:

Condition I.

$$Q^{a_1\alpha_1,\ldots,a_l\alpha_l}_{b_1\beta_1,\ldots,b_m\beta_m}=(u^0_{b_1b_1'}\ldots u^0_{b_mb_m'})(u^0_{a_1a_1'}\ldots u^0_{a_la_l'})^*Q^{a_1'\alpha_1,\ldots,a_l'\alpha_l}_{b_1'\beta_1,\ldots,b_m'\beta_m} \tag{2.3a}$$

$$=Q^{a_1\alpha_1',\ldots,a_l\alpha_l'}_{b_1\beta_1',\ldots,b_m\beta_m'}(u^0_{\beta_1'\beta_1}\ldots u^0_{\beta_m'\beta_m})(u^0_{\alpha_1'\alpha_1}\ldots u^0_{\alpha_l'\alpha_l})^*. \tag{2.3b}$$

Here, summation over repeated indices is assumed.

Equations (2.3) must be fulfilled for arbitrary $\mathbf{u}^0$. The choice

$$u^0_{ab}=\exp(i\vartheta_a)\delta_{ab} \tag{2.4}$$

implies

$$Q^{a_1\alpha_1,\ldots,a_l\alpha_l}_{b_1\beta_1,\ldots,b_m\beta_m}=\exp\{i[(\vartheta_{b_1}+\ldots+\vartheta_{b_m})-(\vartheta_{a_1}+\ldots+\vartheta_{a_l})]\}Q^{a_1\alpha_1,\ldots,a_l\alpha_l}_{b_1\beta_1,\ldots,b_m\beta_m} \tag{2.5a}$$

$$=Q^{a_1\alpha_1,\ldots,a_l\alpha_l}_{b_1\beta_1,\ldots,b_m\beta_m}\exp\{i[(\vartheta_{\beta_1}+\ldots+\theta_{\beta_m})-(\vartheta_{\alpha_1}+\ldots+\vartheta_{\alpha_l})]\}. \tag{2.5b}$$

As the phases ϑ_i are arbitrary, these expressions can only be satisfied if the two sums in the exponent cancel. This is the case if the (non-ordered) sets of upper and lower row indices (the latin indices) coincide, and similarly for the column indices (the greek indices); i.e.

$$\{a\}=\{b\} \qquad \{\alpha\}=\{\beta\}. \tag{2.6}$$

In particular

$$m=l \tag{2.7}$$

is a necessary condition for the Q coefficient to be non-zero. Equations (2.6) and (2.7) were previously obtained by Gaudin and Mello (1981).

From the fact that the matrix elements $u_{c\gamma}$ in (1.9) are commuting c-numbers, we have the next condition:

Condition II.

$$Q^{a_1\alpha_1,\ldots,a_m\alpha_m}_{b_1\beta_1,\ldots,b_m\beta_m} = Q^{a_{1'}\alpha_{1'},\ldots,a_{m'}\alpha_{m'}}_{b_1\beta_1,\ldots,b_m\beta_m} \tag{2.8a}$$

$$= Q^{a_1\alpha_1,\ldots,a_m\alpha_m}_{b_{1'}\beta_{1'},\ldots,b_{m'}\beta_{m'}} \tag{2.8b}$$

where $(1',\ldots,m')$ is any permutation of $(1,\ldots,m)$.

Finally, unitarity of the $\mathbf{u}$ matrices yields the third condition:

Condition III.

$$\sum_{a_1} Q^{a_1\alpha_1,a_2\alpha_2,\ldots,a_m\alpha_m}_{a_1\beta_1,b_2\beta_2,\ldots,b_m\beta_m} = \delta^{\alpha_1}_{\beta_1} Q^{a_2\alpha_2,\ldots,a_m\alpha_m}_{b_2\beta_2,\ldots,b_m\beta_m}. \tag{2.9}$$

Clearly we could have contracted a_i and b_j, giving no new information, because of condition II.

3. The coefficients Q *for* $m=1,2$

We now apply the general conditions I, II, III found in the previous section to the evaluation of the Q coefficients for $m=1$ and $m=2$.

For $m=1$, (2.6) implies that $Q^{a\alpha}_{a\alpha}$ is the only non-vanishing coefficient. Using a permutation for the $\mathbf{u}^0$ of condition I shows that $Q^{a\alpha}_{a\alpha}$ is independent of a and α. Condition II is irrelevant here. Condition III then gives $Q^{a\alpha}_{a\alpha}=1/N$, so that

$$Q^{a\alpha}_{b\beta}=\frac{\delta^a_b\delta^\alpha_\beta}{N}. \tag{3.1}$$

We now go over to $m=2$. We first apply condition I(a), equation (2.3a), to $Q^{11,11}_{11,11}$, to find

$$Q^{11,11}_{11,11}=2\sum_{a\neq b} u^0_{1a}u^0_{1b}(u^0_{1a}u^0_{1b})^* Q^{a1,b1}_{a1,b1}+\sum_{\alpha}|u^0_{1a}|^4 Q^{a1,a1}_{a1,a1}. \tag{3.2}$$

Here we have used (2.6), as well as

$$Q^{a1,b1}_{a1,b1}=Q^{b1,a1}_{a1,b1} \tag{3.3}$$

which follows from condition II.

One can easily see that

$$Q^{a1,b1}_{a1,b1}=Q^{11,21}_{11,21} \qquad a\neq b \tag{3.4a}$$

$$Q^{a1,a1}_{a1,a1}=Q^{11,11}_{11,11} \tag{3.4b}$$

as a result of condition I, where $\mathbf{u}_0$ is chosen as the appropriate permutation. Using (3.4) and the unitarity of $\mathbf{u}^0$, equation (3.2) becomes independent of $\mathbf{u}^0$ and gives

$$Q^{11,11}_{11,11}=2Q^{11,21}_{11,21}. \tag{3.5}$$

From condition III we now have

$$\sum_a Q^{11,a1}_{11,a1}=Q^{11}_{11} \tag{3.6}$$

so that

$$Q^{11,11}_{11,11}+(N-1)Q^{11,21}_{11,21}=1/N. \tag{3.7}$$

Condition I was again used in the second term of (3.7) and result (3.1) on the right-hand side. From (3.5) and (3.7) we then have

$$Q^{11,11}_{11,11}=\frac{2}{N(N+1)} \tag{3.8}$$

$$Q^{11,21}_{11,21}=\frac{1}{N(N+1)}. \tag{3.9}$$

The procedure followed above is the same as the one used by Gaudin and Mello (1981), expressed in the language of the Q, and the present results (3.8) and (3.9) are the same as those given by equations (82) and (83) of that reference.

We can now complete the $m=2$ case, calculating the remaining Q coefficients with a procedure entirely similar to the one we just used. We first apply condition I to $Q^{13,24}_{13,24}$; we use condition II just as above and the result (3.9); $\mathbf{u}^0$ again drops out because of unitarity, giving

$$Q^{13,24}_{13,24}+Q^{23,14}_{13,24}=\frac{1}{N(N+1)}. \tag{3.10}$$

From condition III we obtain

$$\sum_a Q^{a3,24}_{a3,24}=1/N \tag{3.11}$$

or

$$(N-1)Q^{13,24}_{13,24}+Q^{23,24}_{23,24}=1/N \tag{3.12}$$

where we have used condition I in the first term. Employing (3.9) for the second term, we find

$$Q^{13,24}_{13,24}=\frac{1}{(N-1)(N+1)}. \tag{3.13}$$

Equation (3.10) then gives

$$Q^{23,14}_{13,24}=-\frac{1}{(N-1)N(N+1)}. \tag{3.14}$$

Application of condition I with the appropriate permutation for $\mathbf{u}^0$ shows that (3.8), (3.9), (3.13) and (3.14) cover all possibilities for $m=2$.

4. The general structure of the Q coefficients from condition I

From (2.6) we known that $Q^{a_1\alpha_1,\ldots,a_m\alpha_m}_{b_1\beta_1,\ldots,b_m\beta_m}$ must contain terms of the type

$$\Delta^{a_1\ldots a_{m'}}_{b_1\ldots b_m}\Delta^{\alpha_1\ldots\alpha_{m''}}_{\beta_1\ldots\beta_m}\equiv\delta^{a_{1'}}_{b_1}\ldots\delta^{a_{m'}}_{b_m}\delta^{\alpha_{1''}}_{\beta_1}\ldots\delta^{\alpha_{m''}}_{\beta_m} \tag{4.1}$$

where $(1'\ldots m')$ and $(1''\ldots m'')$ are permutations of $(1\ldots m)$.

We show in the appendix that the most general Q satisfying condition I is a linear combination of terms like (4.1); the sum contains terms with all possible permutations of the indices, the coefficients in front being independent of such indices. Here we just prove that a Q so constructed satisfies condition I. The transformation (2.3a) applied to the full Q affects the term containing (4.1) as (repeated indices are summed over)

$$\begin{aligned}(u^0_{b_1\bar{b}_1}\dots u^0_{b_m\bar{b}_m})(u^0_{a_1\bar{a}_1}\dots u^0_{a_m\bar{a}_m})^*\Delta^{\bar{a}_{1'}\dots\bar{a}_{m'}}_{\bar{b}_1\dots\bar{b}_m}\Delta^{\alpha_{1''}\dots\alpha_{m''}}_{\beta_1\dots\beta_m}\\ =(u^0_{b_1\bar{b}_1}\dots u^0_{b_m\bar{b}_m})(u^0_{a_{1'}\bar{a}_{1'}}\dots u^0_{a_{m'}\bar{a}_{m'}})^*\Delta^{\bar{a}_{1'}\dots\bar{a}_{m'}}_{\bar{b}_1\dots\bar{b}_m}\Delta^{\alpha_{1''}\dots\alpha_{m''}}_{\beta_1\dots\beta_m}.\end{aligned}\tag{4.2}$$

To get the second row we have changed the order of the factors occurring in the second group of $\mathbf{u}^0$. This is convenient because, using the unitarity of the matrix $\mathbf{u}^0$ and then employing the definition (4.1) of the Δ, we can write (4.2) as

$$\delta^{a_{1'}}_{b_1}\dots\delta^{a_{m'}}_{b_m}\Delta^{\alpha_{1''}\dots\alpha_{m''}}_{\beta_1\dots\beta_m}=\Delta^{a_{1'}\dots a_{m'}}_{b_1\dots b_m}\Delta^{\alpha_{1''}\dots\alpha_{m''}}_{\beta_1\dots\beta_m}\tag{4.3}$$

which coincides with the original term (4.1) to which we applied the transformation.

A similar analysis can be carried out with the transformation (2.3b). This then proves the statement.

We have thus disposed of condition I. It remains to fulfil conditions II and III. This will be implemented for each particular m that we shall work with. In this paper we consider the cases $m=1,2,3,4$; we shall see that the procedure just outlined gives a *unique* answer, thus indicting that, at least for those cases, conditions I, II, III define the problem uniquely.

As examples, we work out in this section the cases $m=1,2$ that were analysed in the previous section. The cases $m=3$, 4 will be treated in the following sections.

$m=1$. Our ansatz gives

$$Q^{a\alpha}_{b\beta}=A\Delta^a_b\Delta^\alpha_\beta.\tag{4.4}$$

Condition II is irrelevant. Condition III gives

$$\sum_a A\Delta^a_a\Delta^\alpha_\beta=\Delta^\alpha_\beta\tag{4.5}$$

so that $A=1/N$ and

$$Q^{a\alpha}_{b\beta}=\frac{\delta^a_b\delta^\alpha_\beta}{N}\tag{4.6}$$

just as in (3.1).

$m=2$. Our ansatz consists of the linear combination

$$Q^{a\alpha,b\beta}_{a'\alpha',b'\beta'}=A\Delta^{ab}_{a'b'}\Delta^{\alpha\beta}_{\alpha'\beta'}+C\Delta^{ba}_{a'b'}\Delta^{\beta\alpha}_{\alpha'\beta'}+B\Delta^{ab}_{a'b'}\Delta^{\beta\alpha}_{\alpha'\beta'}+D\Delta^{ba}_{a'b'}\Delta^{\alpha\beta}_{\alpha'\beta'}.\tag{4.7}$$

Under condition II(a) (equation (2.8a)), a and b are interchanged *and* α and β are simultaneously interchanged: the first two terms of (4.7) then get interchanged, as well as the last two terms.

Under condition II(b) (equation (2.8b)), a' and b' are interchanged *and* α' and β' are simultaneously interchanged: this operation again interchanges the first two and the last two terms in (4.7). To fulfil condition II we then choose

$$A=C\qquad B=D\tag{4.8}$$

so that

$$Q^{a\alpha,b\beta}_{a'\alpha'b'\beta'}=A(\Delta^{ab}_{a'b'}\Delta^{\alpha\beta}_{\alpha'\beta'}+\Delta^{ba}_{a'b'}\Delta^{\beta\alpha}_{\alpha'\beta'})+B(\Delta^{ab}_{a'b'}\Delta^{\beta\alpha}_{\alpha'\beta'}+\Delta^{ba}_{a'b'}\Delta^{\alpha\beta}_{\alpha'\beta'}).\tag{4.9}$$

Now we have to satisfy condition III (equation (2.9)). We have

$$A(N\delta_{b'}^{b}\Delta_{\alpha'\beta'}^{\alpha\beta}+\delta_{b'}^{b}\Delta_{\alpha'\beta'}^{\beta\alpha})+B(N\delta_{b'}^{b}\Delta_{\alpha'\beta'}^{\beta\alpha}+\delta_{b'}^{b}\Delta_{\alpha'\beta'}^{\alpha\beta})=\delta_{\alpha'}^{\alpha}M_{b'\beta'}^{b\beta}=\delta_{\alpha'}^{\alpha}\frac{\delta_{b'}^{b}\delta_{\beta'}^{\beta}}{N} \tag{4.10}$$

where we have used (4.6) in the last step.

We then get the two linear equations

$$NA+B=\frac{1}{N} \qquad A+NB=0 \tag{4.11}$$

with the solution

$$A=\frac{1}{N^2-1} \qquad B=-\frac{1}{N(N^2-1)}. \tag{4.12}$$

Substituting in (4.9) we then have the answer

$$Q_{a'\alpha',b'\beta'}^{a\alpha_1b\beta}=\frac{1}{N^2-1}(\Delta_{a'b'}^{ab}\Delta_{\alpha'\beta'}^{\alpha\beta}+\Delta_{a'b'}^{ba}\Delta_{\alpha'\beta'}^{\beta\alpha})-\frac{1}{N(N^2-1)}(\Delta_{a'b'}^{ab}\Delta_{\alpha'\beta'}^{\beta\alpha}+\Delta_{a'b'}^{ba}\Delta_{\alpha'\beta'}^{\alpha\beta}). \tag{4.13}$$

We can easily check that the results (3.8), (3.9), (3.13) and (3.14) are particular cases of the general expression (4.13).

In the above analysis we had to consider permutations of the latin indices among themselves and of the greek indices among themselves. Before closing this section we mention some concepts in connection with permutations that will be useful later on.

Consider a collection of indices, their place of occurrence being indicated by a numeral:

$$\begin{array}{llllll} \text{place} & \rightarrow & 1 & 2 & 3 & 4 \quad \ldots \\ \text{index} & \rightarrow & \alpha & \beta & \gamma & \delta \quad \ldots . \end{array} \tag{4.14}$$

Let us denote by (12) a transposition of *places* 1 and 2, and by $(\alpha\beta)$ a transposition of the *indices* α and β and similarly for other numerals and letters. The following two properties will be important for us.

(i) Two place or index transpositions do not commute when they have one element in common.

For instance, under (13) (12) (applied from right to left), $\alpha\beta\gamma\delta$ becomes $\gamma\alpha\beta\delta$, while under (12) (13) it becomes $\beta\gamma\alpha\delta$.

(ii) A place and an index transposition commute.

For instance, under (12) $(\alpha\beta)$, or $(\alpha\beta)(12)$, $\alpha\beta\gamma\delta$ becomes $\alpha\beta\gamma\delta$; under (23) $(\alpha\beta)$, or $(\alpha\beta)(23)$, it becomes $\beta\gamma\alpha\delta$; under (34) $(\alpha\beta)$, or $(\alpha\beta)$ (34), $\alpha\beta\gamma\delta$ becomes $\beta\alpha\delta\gamma$.

We shall use below the notation $(\alpha\beta\gamma\ldots\delta)$ to denote the cyclic *index* permutation that replaces α by β, β by $\gamma,\ldots,\delta$ by α; the notation $(1\,2\,3\ldots 4)$ denotes the cyclic *place* permutation that takes the letter in place 1 to place 2, the one in place 2 to place 3, . . . , the one in place 4 to place 1.

5. The Q coefficients for $m=3$

We calculate in this section the coefficient

$$Q_{a'\alpha',b'\beta',c'\gamma'}^{a\alpha,b\beta,c\gamma} \tag{5.1}$$

which, according to the discussion at the beginning of section 4, we write as a linear combination of $3!3!=36$ terms, obtained from

$$\Delta_{a'b'c'}^{abc}\Delta_{\alpha'\beta'\gamma'}^{\alpha\beta\gamma} \tag{5.2}$$

by leaving the lower indices as they stand and permuting a, b, c in 3! ways and α, β, γ in 3! ways. We organise the 36 terms as follows.

First, consider the six terms

$$\begin{array}{ccc} & [ab]\ (\alpha\beta) & [ac]\ (\alpha\gamma) \\ [abc](\alpha\beta\gamma) & [bac](\beta\alpha\gamma) & [cba](\gamma\beta\alpha) \\ [bc]\ (\beta\gamma) & [abc](\alpha\beta\gamma) & [acb](\alpha\gamma\beta) \\ [acb](\alpha\gamma\beta) & [bca](\beta\gamma\alpha) & [cab](\gamma\alpha\beta). \end{array} \tag{5.3}$$

Each terms is indicated just by the upper indices of each Δ, since we agreed to leave the lower indices as in (5.2); square brackets enclose latin indices, and round brackets, greek indices. For instance, the first term $[abc](\alpha\beta\gamma)$ in (5.3) is just (5.2). Above each term in (5.3) we have indicated the index permutation that applied to $[abc](\alpha\beta\gamma)$ gives the term in question. Notice that each permutation of latin indices is accompanied by the same permutation of greek indices: this corresponds to a permutation of the $u^*_{a\alpha}$ in the Q coefficient in (1.9), as in (2.8a). Thus the operation (2.8a) applied to one term in (5.3) generates the other five. Condition II(a) thus implies that *the six terms (5.3) must have the same coefficient.*

We now apply to the greek indices of (5.3) the *place* transpositions (12), (13), (23), to get, respectively ($[e]$ denotes the unit element applied to the latin indices):

$$[e](12)\Rightarrow\begin{cases}[abc](\beta\alpha\gamma) & [bac](\alpha\beta\gamma) \quad [cba](\beta\gamma\alpha) \\ [acb](\gamma\alpha\beta) & [bca](\gamma\beta\alpha) \quad [cab](\alpha\gamma\beta)\end{cases} \tag{5.4}$$

$$[e](13)\Rightarrow\begin{cases}[abc](\gamma\beta\alpha) & [bac](\gamma\alpha\beta) \quad [cba](\alpha\beta\gamma) \\ [acb](\beta\gamma\alpha) & [bca](\alpha\gamma\beta) \quad [cab](\beta\alpha\gamma)\end{cases} \tag{5.5}$$

$$[e](23)\Rightarrow\begin{cases}[abc](\alpha\gamma\beta) & [bac](\beta\gamma\alpha) \quad [cba](\gamma\alpha\beta) \\ [acb](\alpha\beta\gamma) & [bca](\beta\alpha\gamma) \quad [cab](\gamma\beta\alpha).\end{cases} \tag{5.6}$$

We can easily check that if we apply the *index* permutations indicated in (5.3) to one of the terms in (5.4), we generate the other five terms; a similar fact occurs with (5.5) and (5.6). For instance, $[ac](\alpha\gamma)$ (which, applied to the first term in (5.3), gives the third term), applied to the first term in (5.4) gives the third term of (5.4). This is easy to understand: the operation in question can be thought of as, first, applying to the first term in (5.3) the place transposition $[e](12)$ and, next, the index transposition $[ac](\alpha\gamma)$; since place and index transpositions commute (see the end of last section), we could have first applied $[ac](\alpha\gamma)$ to the first term in (5.3) to get the third one; application of $[e](12)$ then gives (according to the way in which (5.4) was constructed) precisely the third term of (5.4).

As we mentioned above, each of the permutations indicated in (5.3) corresponds to an operation associated with condition II(a); that condition thus implies that *the six coefficients in (5.4) must have the same coefficients, and similarly for (5.5) and (5.6).*

We can also interpret each term in (5.3) as being obtained from the first one by applying the *place* permutation indicated on top of the corresponding term in the following sequence

$$\begin{array}{ccc} & [12]\ (12) & [13]\ (13) \\ [abc](\alpha\beta\gamma) & [bac](\beta\alpha\gamma) & [cba](\gamma\beta\alpha) \\ [23]\ (23) & [132]\,(132) & [123]\,(123) \\ [acb](\alpha\gamma\beta) & [bca](\beta\gamma\alpha) & [cab](\gamma\alpha\beta). \end{array} \tag{5.7}$$

The *place* permutations indicated above are the same for latin as for greek indices, so they are equivalent to the same permutation of the latin and of the greek *lower* indices of the Δ in (5.2); they thus correspond to a permutation of the $u_{a\alpha}$ in (1.9), and hence to one of the operations (2.8b): any one of these operations applied to a term in (5.7) generates the other five. Condition II(b) thus implies the equality of the corresponding coefficients, which we already found in connection with property II(a).

On the other hand, it is easy to check that if we apply all the *place* permutations indicated in (5.7) to one of the terms in (5.4), we do not stay inside (5.4), but get terms belonging to (5.4), (5.5) and (5.6). For instance, application of [13](13) to the first term in (5.4) gives the third term in (5.6). This is easy to understand: we can think of first applying [e](12) to the first element of (5.7), followed by [13](13); if these two operations commuted, we could apply [13](13) first, to get the third term in (5.7), and then [e](12), to get the third term in (5.4), and we would not get outside of (5.4); because of the lack of commutativity we have, instead, [13](13)[e](12) = [e](23)[13](13), which first takes us to the third term in (5.7), and from there to the third one in (5.6). Non-commutativity, in general, of place permutations, is thus responsible for this fact. We found above just one coefficient for the six terms (5.4), one for (5.5) and one for (5.6); condition II(b) now implies the equality of the three coefficients.

Finally, we apply to the greek indices of (5.3) the *place* cyclic permutations (123) and (132) (see the end of the previous section), to get, respectively

$$[e](123) \Rightarrow \begin{cases} [abc](\gamma\alpha\beta) & [bac](\gamma\beta\alpha) & [cba](\alpha\gamma\beta) \\ [acb](\beta\alpha\gamma) & [bca](\alpha\beta\gamma) & [cab](\beta\gamma\alpha) \end{cases} \tag{5.8}$$

$$[e](132) \Rightarrow \begin{cases} [abc](\beta\gamma\alpha) & [bac](\alpha\gamma\beta) & [cba](\beta\alpha\gamma) \\ [acb](\gamma\beta\alpha) & [bca](\gamma\alpha\beta) & [cab](\alpha\beta\gamma). \end{cases} \tag{5.9}$$

Again, operation II(a) generates the various terms inside (5.8) or (5.9), while II(b) mixes (5.8) and (5.9). *All the coefficients of the terms in (5.8) and (5.9) are thus the same.*

Expressions (5.3)-(5.6), (5.8), (5.9) contain the 36 terms needed. The linear combination that satisfies conditions I and II is thus (in the notation explained directly following (5.3))

$$\begin{aligned} Q^{a\alpha,b\beta,c\gamma}_{a'\alpha',b'\beta',c'\gamma'} = {} & A\{[abc](\alpha\beta\gamma)+[bac](\beta\alpha\gamma)+[cba](\gamma\beta\alpha) \\ & +[acb](\alpha\gamma\beta)+[bca](\beta\gamma\alpha)+[cab](\gamma\alpha\beta)\} \\ & + B\{[abc](\beta\alpha\gamma)+[bac](\alpha\beta\gamma)+[cba](\beta\gamma\alpha) \\ & +[acb](\gamma\alpha\beta)+[bca](\gamma\beta\alpha)+[cab](\alpha\gamma\beta) \\ & +[abc](\gamma\beta\alpha)+[bac](\gamma\alpha\beta)+[cba](\alpha\beta\gamma) \\ & +[acb](\beta\gamma\alpha)+[bca](\alpha\gamma\beta)+[cab](\beta\alpha\gamma) \\ & +[abc](\alpha\gamma\beta)+[bac](\beta\gamma\alpha)+[cba](\gamma\alpha\beta) \\ & +[acb](\alpha\beta\gamma)+[bca](\beta\alpha\gamma)+[cab](\gamma\beta\alpha)\} \\ & + C\{[abc](\gamma\alpha\beta)+[bac](\gamma\beta\alpha)+[cba](\alpha\gamma\beta) \\ & +[acb](\beta\alpha\gamma)+[bca](\alpha\beta\gamma)+[cab](\beta\gamma\alpha) \\ & +[abc](\beta\gamma\alpha)+[bac](\alpha\gamma\beta)+[cba](\beta\alpha\gamma) \\ & +[acb](\gamma\beta\alpha)+[bca](\gamma\alpha\beta)+[cab](\alpha\beta\gamma)\}. \end{aligned} \tag{5.10}$$

We now enforce condition III, equation (2.9); i.e.

$$\sum_a Q^{a\alpha,b\beta,c\gamma}_{a\alpha',b'\beta',c'\gamma'} = \delta^{\alpha}_{\alpha'} Q^{b\beta,c\gamma}_{b'\beta',c'\gamma'}. \tag{5.11}$$

We use (5.10) on the LHS and (4.13) on the RHS, to get

$$\begin{aligned}
&A\{N[bc](\alpha\beta\gamma)+[bc](\beta\alpha\gamma)+[bc](\gamma\beta\alpha)\\
&\quad+N[cb](\alpha\gamma\beta)+[cb](\beta\gamma\alpha)+[cb](\gamma\alpha\beta)\}\\
&+B\{N[bc](\beta\alpha\gamma)+[bc](\alpha\beta\gamma)+[bc](\beta\gamma\alpha)\\
&\quad+N[cb](\gamma\alpha\beta)+[cb](\gamma\beta\alpha)+[cb](\alpha\gamma\beta)\\
&\quad+N[bc](\gamma\beta\alpha)+[bc](\gamma\alpha\beta)+[bc](\alpha\beta\gamma)\\
&\quad+N[cb](\beta\gamma\alpha)+[cb](\alpha\gamma\beta)+[cb](\beta\alpha\gamma)\\
&\quad+N[bc](\alpha\gamma\beta)+[bc](\beta\gamma\alpha)+[bc](\gamma\alpha\beta)\\
&\quad+N[cb](\alpha\beta\gamma)+[cb](\beta\alpha\gamma)+[cb](\gamma\beta\alpha)\}\\
&+C\{N[bc](\gamma\alpha\beta)+[bc](\gamma\beta\alpha)+[bc](\alpha\gamma\beta)\\
&\quad+N[cb](\beta\alpha\gamma)+[cb](\alpha\beta\gamma)+[cb](\beta\gamma\alpha)\\
&\quad+N[bc](\beta\gamma\alpha)+[bc](\alpha\gamma\beta)+[bc](\beta\alpha\gamma)\\
&\quad+N[cb](\gamma\beta\alpha)+[cb](\gamma\alpha\beta)+[cb](\alpha\beta\gamma)\}\\
&=\frac{1}{N^2-1}\{[bc](\alpha\beta\gamma)+[cb](\alpha\gamma\beta)\}-\frac{1}{N(N^2-1)}\\
&\quad\times\{[bc](\alpha\gamma\beta)+[cb](\alpha\beta\gamma)\}.
\end{aligned}\tag{5.12}$$

In (5.12) the Δ associated with the latin indices has lower indices $b'c'$, and that associated with the greek indices has lower indices $\alpha'\beta'\gamma'$.

Equating, in (5.12), the coefficients of the various Δ, we get the four linear equations

$$NA+2B=\frac{1}{N^2-1} \tag{5.13a}$$

$$A+NB+C=0 \tag{5.13b}$$

$$2B+NC=0 \tag{5.13c}$$

$$NB+2C=-\frac{1}{N(N^2-1)} \tag{5.13d}$$

of which only three are independent: indeed, substituting A from (5.13b) and B from (5.13c) into (5.13a), we obtain (5.13d). The solution of (5.13a)-(5.13c) is

$$A=\frac{N^2-2}{N(N^2-1)(N^2-4)} \tag{5.14}$$

$$B=-\frac{1}{(N^2-1)(N^2-4)} \tag{5.15}$$

$$C=\frac{2}{N(N^2-1)(N^2-4)}. \tag{5.16}$$

Equations (5, 10), (5, 14)-(5, 16) are the solution to our problem for $m=3$.

6. The Q coefficient for $m=4$

We calculate in this section the coefficient

$$Q^{a\alpha,b\beta,c\gamma,d\delta}_{a'\alpha',b'\beta',c'\gamma',d'\delta'}. \tag{6.1}$$

According to section 4, we write (6.1) as a linear combination of 4!4! terms, obtained from

$$\Delta^{abcd}_{a'b'c'd'}\Delta^{\alpha\beta\gamma\delta}_{\alpha'\beta'\gamma'\delta'} \tag{6.2}$$

by leaving the lower indices as in (6.2) and permuting $abcd$ in 4! ways and $\alpha\beta\gamma\delta$ in 4! ways. The linear combination that satisfies conditions I and II can now be written as (in the notation explained directly following (5.3))

$$\begin{aligned}
Q^{a\alpha,b\beta,c\gamma,d\delta}_{a'\alpha',b'\beta',c'\gamma',d'\delta'}
&= A\{[abcd](\alpha\beta\gamma\delta)+[abdc](\alpha\beta\delta\gamma)+[acbd](\alpha\gamma\beta\delta)\\
&\quad +[acdb](\alpha\gamma\delta\beta)+[adbc](\alpha\delta\beta\gamma)+[adcb](\alpha\delta\gamma\beta)\\
&\quad +[bacd](\beta\alpha\gamma\delta)+[badc](\beta\alpha\delta\gamma)+[bcad](\beta\gamma\alpha\delta)\\
&\quad +[bcda](\beta\gamma\delta\alpha)+[bdac](\beta\delta\alpha\gamma)+[bdca](\beta\delta\gamma\alpha)\\
&\quad +[cabd](\gamma\alpha\beta\delta)+[cadb](\gamma\alpha\delta\beta)+[cbad](\gamma\beta\alpha\delta)\\
&\quad +[cbda](\gamma\beta\delta\alpha)+[cdab](\gamma\delta\alpha\beta)+[cdba](\gamma\delta\beta\alpha)\\
&\quad +[dabc](\delta\alpha\beta\gamma)+[dacb](\delta\alpha\gamma\beta)+[dbac](\delta\beta\alpha\gamma)\\
&\quad +[dbca](\delta\beta\gamma\alpha)+[dcab](\delta\gamma\alpha\beta)+[dcba](\delta\gamma\beta\alpha)\}\\
&\quad + B\{[abcd](\beta\alpha\gamma\delta)+\ldots+[dcba](\gamma\delta\beta\alpha)\\
&\quad +[abcd](\gamma\beta\alpha\delta)+\ldots+[dcba](\beta\gamma\delta\alpha)\\
&\quad +[abcd](\delta\beta\gamma\alpha)+\ldots+[dcba](\alpha\gamma\beta\delta)\\
&\quad +[abcd](\alpha\gamma\beta\delta)+\ldots+[dcba](\delta\beta\gamma\alpha)\\
&\quad +[abcd](\alpha\delta\gamma\beta)+\ldots+[dcba](\delta\alpha\beta\gamma)\\
&\quad +[abcd](\alpha\beta\delta\gamma)+\ldots+[dcba](\delta\gamma\alpha\beta)\}\\
&\quad + C\{[abcd](\gamma\alpha\beta\delta)+\ldots+[dcba](\beta\delta\gamma\alpha)\\
&\quad +[abcd](\beta\gamma\alpha\delta)+\ldots+[dcba](\gamma\beta\delta\alpha)\\
&\quad +[abcd](\delta\alpha\gamma\beta)+\ldots+[dcba](\alpha\delta\beta\gamma)\\
&\quad +[abcd](\beta\delta\gamma\alpha)+\ldots+[dcba](\gamma\alpha\beta\delta)\\
&\quad +[abcd](\delta\beta\alpha\gamma)+\ldots+[dcba](\alpha\gamma\delta\beta)\\
&\quad +[abcd](\gamma\beta\delta\alpha)+\ldots+[dcba](\beta\gamma\alpha\delta)\\
&\quad +[abcd](\alpha\delta\beta\gamma)+\ldots+[dcba](\delta\alpha\gamma\beta)\\
&\quad +[abcd](\alpha\gamma\delta\beta)+\ldots+[dcba](\delta\beta\alpha\gamma)\}\\
&\quad + D\{[abcd](\beta\alpha\delta\gamma)+\ldots+[dcba](\gamma\delta\alpha\beta)\\
&\quad +[abcd](\gamma\delta\alpha\beta)+\ldots+[dcba](\beta\alpha\delta\gamma)\\
&\quad +[abcd](\delta\gamma\beta\alpha)+\ldots+[dcba](\alpha\beta\gamma\delta)\}\\
&\quad + E\{[abcd](\delta\alpha\beta\gamma)+\ldots+[dcba](\alpha\delta\gamma\beta)\\
&\quad +[abcd](\gamma\alpha\delta\beta)+\ldots+[dcba](\beta\delta\alpha\gamma)\\
&\quad +[abcd](\delta\gamma\alpha\beta)+\ldots+[dcba](\alpha\beta\delta\gamma)\\
&\quad +[abcd](\beta\delta\alpha\gamma)+\ldots+[dcba](\gamma\alpha\delta\beta)\\
&\quad +[abcd](\gamma\delta\beta\alpha)+\ldots+[dcba](\beta\alpha\gamma\delta)\\
&\quad +[abcd](\beta\gamma\delta\alpha)+\ldots+[dcba](\gamma\beta\alpha\delta)\}.
\end{aligned} \tag{6.3}$$

In each line of (6.3) containing an ellipsis (...) there are 24 terms (of which only the first and last one are shown), obtained by applying one of the following place permutations to the greek indices of the 24 terms that multiply the coefficient A: (12), (13), (14), (23), (24), (34), to obtain, respectively, each of the six lines that multiply the coefficient B; (123), (132), (124), (142), (134), (143), (234), (243), respectively, for the eight lines that multiply the coefficient C; (12)(34), (13)(24), (14)(23), for the three lines that multiply the coefficient D; (1234) (1243), (1324), (1342), (1423), (1432), for the six lines that multiply the coefficient E.

Just as in the previous section, a transformation associated with condition II(a), equation (2.8a), transforms among themselves the 24 terms inside the curly bracket that multiplies A in the above equation, as well the 24 terms inside each one of subsequent rows; a transformation associated with condition II(b), equation (2.8b), again transforms among themselves the terms inside the curly bracket that multiplies A, but it also mixes the various rows contained in each of the subsequent curly brackets. The net result is that we have five different coefficients (A, B, C, D, E), that have to be fixed using condition III (unitarity), equation (2.9).

We thus insert (6.3) on the LHS of (2.9) and (5.10) on the RHS. Equating the coefficients of the various Δ, we obtain the linear equations

$$AN+3B=\frac{N^2-2}{N(N^2-1)(N^2-4)} \tag{6.4a}$$

$$A+NB+2C=0 \tag{6.4b}$$

$$2B+NC+E=0 \tag{6.4c}$$

$$B+ND+2E=0 \tag{6.4d}$$

$$2C+D+NE=0 \tag{6.4e}$$

$$NB+2C+D=-\frac{1}{(N^2-1)(N^2-4)} \tag{6.4f}$$

$$NC+3E=\frac{2}{N(N^2-1)(N^2-4)}. \tag{6.4g}$$

Equations (6.4a)-(6.4e) constitute five independent relations among the coefficients A, B, C, D, E, whose solution is

$$A=\frac{N^4-8N^2+6}{N^2(N^2-1)(N^2-4)(N^2-9)} \tag{6.5}$$

$$B=-\frac{1}{N(N^2-1)(N^2-9)} \tag{6.6}$$

$$C=\frac{2N^2-3}{N^2(N^2-1)(N^2-4)(N^2-9)} \tag{6.7}$$

$$D=\frac{N^2+6}{N^2(N^2-1)(N^2-4)(N^2-9)} \tag{6.8}$$

$$E=-\frac{5}{N(N^2-1)(N^2-4)(N^2-9)}. \tag{6.9}$$

One can easily check that the above solution satisfies the remaing equations (6.4f), (6.4g).

Inserting (6.5)-(6.9) in (6.3) we have the expression for the Q coefficient for $m=4$.

7. Applications to the problem of disordered conductors

In this section we summarise the physical consequences found by Mello *et al* (1988a) and Mello and Stone (1990) using the results derived in the previous sections.

From (1.10) and (3.1) we find, for the average of the transmission coefficient

$$\langle T_{ab}\rangle_L^{(\beta)} = \frac{\langle T\rangle_L^{(\beta)}}{N^2} \tag{7.1}$$

where $T = \Sigma_{ab} T_{ab}$ is the total transmission coefficient into all channels, when the incident ones are fed with N incoherent fluxes; T can also be identified with the conductance g of the system measured in units of e^2/h, for 'spinless electrons' (Fisher and Lee 1981).

Using (4.13) we can calculate the crossed second moment (1.11) of the transmission coefficients; we then get, for the covariance

$$C^T_{ab,a'b'} = \langle T_{ab}T_{a'b'}\rangle_L^{(\beta)} - \langle T_{ab}\rangle_L^{(\beta)}\langle T_{a'b'}\rangle_L^{(\beta)} \tag{7.2}$$

the expression

$$\begin{aligned} C^T_{ab,a'b'} = {} & [A_N\langle T^2\rangle_L^{(\beta)} - B_N\langle T_2\rangle_L^{(\beta)}]\delta_{aa'}\delta_{bb'} + [A_N\langle T_2\rangle_L^{(\beta)} - B_N\langle T^2\rangle_L^{(\beta)}](\delta_{aa'} + \delta_{bb'}) \\ & + [A_N\langle T^2\rangle_L^{(\beta)} - N^2 B_N\langle T_2\rangle_L^{(\beta)} - C_N\langle T^2\rangle_L^{(\beta)}]. \end{aligned} \tag{7.3}$$

Here we have defined

$$\begin{aligned} T_k &= \sum_a \frac{1}{(1+\lambda_a)^k} & \qquad A_N &= \frac{N^2+1}{N^2(N^2-1)^2} \\ B_N &= \frac{2}{N^3(N^2-1)^2} & C_N &= \frac{1}{N^4}. \end{aligned} \tag{7.4}$$

Equation (7.3) is exact. As a check, we can easily verify that the sum of (7.3) over a, b, a', b' gives precisely var T. The structure of (7.3) is the same for $\beta = 1, 2$, although the value of the coefficients of the δ functions does depend on the specific value of β.

Feng *et al* (1988) in their equation (3) also obtained three types of terms for the covariance (7.3): setting $W \ll L$ (quasi-1D systems) they are seen to have essentially the structure provided by the δ functions of (7.3). The difference is that our Kronecker deltas are replaced by functions which peak at those wave-vectors which satisfy the appropriate Kronecker deltas in our calculation, but decay over some distance in momentum space.

To be more specific about the coefficients of the δ functions in (7.3), we now use the results of Mello (1988) and Mello and Stone (1990). To leading order in $N \gg 1$ and for $L/l \gg 1$ one finds

$$C^T_{ab,a'b'} = \langle T_{ab}\rangle_L\langle T_{a'b'}\rangle_L\left(\delta_{aa'}\delta_{bb'} + \frac{2}{3\langle T\rangle_L}(\delta_{aa'} + \delta_{bb'}) + \frac{1+\delta_{\beta 1}}{15[\langle T\rangle_L]^2}\right). \tag{7.5}$$

Here we have dropped the index β in $\langle T_{ab}\rangle_L$ and $\langle T\rangle_L$, because the leading term is independent of β. Equation (7.5) is consistent with (3) of Feng *et al* (1988).

Summing (7.5) over a, b, a', b', we find

$$\operatorname{var} T = \frac{1+\delta_{\beta 1}}{15} \tag{7.6}$$

a result that is independent of the number N of channels, the length L of the conductor and the elastic mean free path l (universal conductance fluctuations).

We now turn to a study of the reflection coefficient R_{ab}.

Using (4.13) we obtain, for the average reflection coefficient (1.12), the expressions

$$\langle R_{ab}\rangle_L^{(\beta)} = \begin{cases} (1+\delta_{ab})\dfrac{\langle R\rangle_L^{(\beta=1)}}{N(N+1)} & \beta=1 \qquad (7.7a) \\[2ex] \dfrac{\langle R\rangle_L^{(\beta=2)}}{N^2} & \beta=2. \qquad (7.7b) \end{cases}$$

Equation (7.7a) means that, with time-reversal symmetry ($\beta=1$), backward scattering to the same channel is enhanced by a factor of 2 as compared with scattering to any other channel. This is precisely the prediction of weak-localisation theory, where one finds that the various paths contribute with random phases, except for a path and its time-reversed one, which contribute coherently and give rise to a factor 2 in the backward direction.

Using (4.13) and (6.3) we can now calculate the crossed second moment (1.13) of the reflection coefficients, and then the covariance

$$C^{R}_{ab,a'b'} = \langle R_{ab}R_{a'b'}\rangle_L^{(\beta)} - \langle R_{ab}\rangle_L^{(\beta)}\langle R_{a'b'}\rangle_L^{(\beta)}. \tag{7.8}$$

To leading order in N and in the limit $L/l \gg 1$ we get

$$C^{R,\beta=1}_{ab,a'b'} = \langle R_{ab}\rangle_L\langle R_{a'b'}\rangle_L\bigg(\frac{\delta_{aa'}\delta_{bb'}+\delta_{ab'}\delta_{a'b}}{1+\delta_{ab}} + \frac{1}{\langle R\rangle_L} \times(\delta_{ab}\delta_{a'b'}\delta_{aa'} - \delta_{aa'} - \delta_{bb'} - \delta_{ab'} - \delta_{ba'}) + \frac{32}{15\langle R\rangle_L^2}\bigg) \tag{7.9a}$$

$$C^{R,\beta=2}_{ab,a'b'} = \langle R_{ab}\rangle_L\langle R_{a'b'}\rangle_L\bigg(\delta_{aa'}\delta_{bb'} - \frac{\delta_{aa'}+\delta_{bb'}}{\langle R\rangle_L} + \frac{16}{15\langle R\rangle_L^2}\bigg). \tag{7.9b}$$

We can easily verify that the sum of (7.9) over a, b, a', b' gives $\operatorname{var} R = \operatorname{var} T$ as given by (7.6).

It would be very nice if one could measure, probably in optical experiments, the correlations studied above, which, as we saw, have expressions that are far from obvious.

8. Summary and conclusions

The purpose of the present paper was to calculate the geometrical Q coefficients of (1.9) up to the order required by the maximum-entropy theory of disordered conductors to compute averages and covariances of transmission and reflection coefficients; the physical consequences for these latter quantities were also briefly discussed.

The Q coefficients must satisfy conditions I, II and III found in section 2, which arise, basically, from the invariance of the Haar measure, the fact that the matrix elements $u_{a\alpha}$ are commuting c-numbers and unitarity of the matrix $\mathbf{u}$. We found the most general structure of the Q coefficients in order to fulfil condition I, arising from the invariance property of the measure. Conditions II and III were then imposed subsequently for the cases $m=1$, 2, 3, 4 and shown to yield a unique answer to the problem.

The Q coefficients calculated in sections 2-6 are used in section 7 to find averages and covariances of transmission and reflection coefficients. Results are consistent with the enhanced backscattering predicted by weak-localisation theory; we also found, for the covariance of the transmission coefficients, results that, in the quasi-1D limit, are consistent with microscopic diagrammatic calculations.

Finally, we remark that the case for which the method used here was originally devised (Mello and Seligman 1980) dealt with unitary and symmetric matrices, which do not form a group. In contrast, the matrices $\mathbf{u}$ of (1.9) are not restricted by the condition of symmetry, so that they form the unitary group $U(N)$. It is thus conceivable that our results could be obtained, perhaps in a simpler way, in terms of the coupling coefficients of $U(N)$, using the fact that integrals over matrix elements of irreducible representations of compact groups with the Haar measure are quite trivial. This alternative procedure would be worth investigating.

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Appendix

We prove here the assertion made at the beginning of section 4 that the most general $Q^{a_1\alpha_1,\ldots,a_m\alpha_m}_{b_1\beta_1,\ldots,b_m\beta_m}$ satisfying condition I is a linear combination with constant coefficients of terms like (4.1).

Consider, to begin with, the simplified object $q^{a_1\ldots a_m}_{b_1\ldots b_m}$, that satisfies the transformation law

$$q^{a_1\ldots a_m}_{b_1\ldots b_m}=(u_{b_1\bar{b}_1}\ldots u_{b_m\bar{b}_m})(u_{a_1\bar{a}_1}\ldots u_{a_m\bar{a}_m})^*q^{\bar{a}_1\ldots\bar{a}_m}_{\bar{b}_1\ldots\bar{b}_m}. \tag{A.1}$$

We shall prove that the most general q satisfying (A.1) is a linear combination, with constant coefficients, of terms like $\Delta^{a_1\ldots a_{m'}}_{b_1\ldots b_m}$. We first work out the cases $m=1,2,3$ and then present the generalisation to arbitrary m.

$m=1$. Equation (A.1) now has the form

$$q^a_b=u_{b\bar{b}}u^*_{a\bar{a}}q^{\bar{a}}_{\bar{b}}. \tag{A.2}$$

Equation (2.6) implies for q^a_b the form

$$q^a_b=A_a\delta^a_b. \tag{A.3}$$

Condition (A.2), with $\mathbf{u}$ chosen as

$$
\mathbf{u} = \begin{array}{cc}
 & \begin{array}{ccccc} & i & & j & \end{array}\\
\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} &
\left[\begin{array}{c:c:c:c:c}
\begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & & 0 & & 0\\
\hdashline
 & 0 & & 1 & \\
\hdashline
0 & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & & 0\\
\hdashline
 & 1 & & 0 & \\
\hdashline
0 & & 0 & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix}
\end{array}\right]
\end{array}
\tag{A.4}
$$

shows that $A_i = A_j$, so that

$$q^a_b = A\delta^a_b \tag{A.5}$$

which proves the statement.

We notice that (A.5) is nothing but Schur's lemma of group theory. In fact, (A.2) can be written in matrix notation as

$$\mathbf{q} = \mathbf{uqu}^\dagger \tag{A.6}$$

or

$$\mathbf{qu} = \mathbf{uq}. \tag{A.7}$$

Schur's lemma states that a matrix that commutes with all the matrices of an irreducible representation (which is the case here, because the $N \times N$ matrices $\mathbf{u}$ are the defining matrices of $\mathrm{U}(N)$) must be a multiple of the unit matrix: this is our (A.5) above.

m = 2. From (2.6), q must be of the form

$$q^{a_1a_2}_{b_1b_2} = A_{b_1b_2}\Delta^{a_1a_2}_{b_1b_2} + B_{b_1b_2}\Delta^{a_2a_1}_{b_1b_2}. \tag{A.8}$$

Let $b_1 \neq b_2$. Consider $a_1 = b_1$, $a_2 = b_2$. Then $q^{b_1b_2}_{b_1b_2} = A_{b_1b_2}$. From (A.1) with the appropriate $\mathbf{u}$ (like (A.4)) we can show that $q^{b_1\neq b_2}_{b_1\neq b_2}$ is independent of b_1, b_2, so that $A_{b_1\neq b_2} = A$. Similarly, $B_{b_1\neq b_2} = B$. For arbitrary b_1, b_2 we then have

$$q^{a_1a_2}_{b_1b_2} = A\Delta^{a_1a_2}_{b_1b_2} + B\Delta^{a_2a_1}_{b_1b_2} + C_{b_1}\delta_{b_1b_2}\Delta^{a_1a_2}_{b_1b_1}. \tag{A.9}$$

Equation (A.1) shows that Q^{bb}_{bb} is independent of b, so that $C_b = C$; i.e.

$$q^{a_1a_2}_{b_1b_2} = A\Delta^{a_1a_2}_{b_1b_2} + B\Delta^{a_2a_1}_{b_1b_2} + C\delta_{b_1b_2}\Delta^{a_1a_2}_{b_1b_1}. \tag{A.10}$$

We now apply (A.1) to (A.10). The first two terms of (A.10) cancel, because they satisfy (A.1) identically, according to the general proof at the beginning of section 4. We then have

$$C\delta_{b_1b_2}\Delta^{a_1a_2}_{b_1b_1} = C\sum_c u_{a_1c}u_{a_2c}(u_{b_1c}u_{b_2c})^*. \tag{A.11}$$

For example, for

$$a_1 = a_2 = b_1 = b_2 = 1 \tag{A.12}$$

(A.11) becomes

$$C = C\sum_c |u_{1c}|^4. \tag{A.13}$$

Since (A.13) must be satisfied for an arbitrary unitary matrix, we obtain $C=0$. Thus, for $m=2$,

$$q^{a_1a_2}_{b_1b_2} = A\Delta^{a_1a_2}_{b_1b_2} + B\Delta^{a_2a_1}_{b_1b_2}$$

is the most general q satisfying (A.1).

m = 3. An analysis similar to the one that led to (A.10) shows that, for $m=3$, q must be of the form

$$\begin{aligned} q^{a_1a_2a_3}_{b_1b_2b_3} = {} & A\Delta^{a_1a_2a_3}_{b_1b_2b_3} + B\Delta^{a_2a_1a_3}_{b_1b_2b_3} + C\Delta^{a_3a_2a_1}_{b_1b_2b_3} \\ & + D\Delta^{a_1a_3a_2}_{b_1b_2b_3} + E\Delta^{a_2a_3a_1}_{b_1b_2b_3} + F\Delta^{a_3a_1a_2}_{b_1b_2b_3} \\ & + \delta_{b_1b_2}(G\Delta^{a_1a_2a_3}_{b_1b_1b_3} + H\Delta^{a_2a_3a_1}_{b_1b_1b_3} + I\Delta^{a_1a_3a_2}_{b_1b_1b_3}) \\ & + \delta_{b_1b_3}(J\Delta^{a_1a_2a_3}_{b_1b_2b_1} + K\Delta^{a_2a_1a_3}_{b_1b_2b_1} + L\Delta^{a_1a_3a_2}_{b_1b_2b_1}) \\ & + \delta_{b_2b_3}(M\Delta^{a_1a_2a_3}_{b_1b_2b_2} + N\Delta^{a_2a_1a_3}_{b_1b_2b_2} + P\Delta^{a_3a_1a_2}_{b_1b_2b_2}) \\ & + \delta_{b_1b_2}\delta_{b_2b_3}R\Delta^{a_1a_2a_3}_{b_1b_1b_1}. \end{aligned} \tag{A.14}$$

In applying (A.1) to (A.14) we recall that the first six terms of (A.14) satisfy (A.1) identically, so that we have

$$\begin{aligned} & \delta_{b_1b_2}(G\Delta^{a_1a_2a_3}_{b_1b_1b_3} + H\Delta^{a_2a_3a_1}_{b_1b_1b_3} + I\Delta^{a_1a_3a_2}_{b_1b_1b_3}) \\ & \quad + \delta_{b_1b_3}(J\Delta^{a_1a_2a_3}_{b_1b_2b_1} + K\Delta^{a_2a_1a_3}_{b_1b_2b_1} + L\Delta^{a_1a_3a_2}_{b_1b_2b_1}) \\ & \quad + \delta_{b_2b_3}(M\Delta^{a_1a_2a_3}_{b_1b_2b_2} + N\Delta^{a_2a_1a_3}_{b_1b_2b_2} + P\Delta^{a_3a_1a_2}_{b_1b_2b_2}) \\ & \quad + \delta_{b_1b_2}\delta_{b_2b_3}R\Delta^{a_1a_2a_3}_{b_1b_1b_1} \\ & = \sum_c [Gu_{a_1c}u_{a_2c}(u_{b_1c}u_{b_2c})^*\delta^{a_3}_{b_3} + Hu_{a_2c}u_{a_3c}(u_{b_1c}u_{b_2c})^*\delta^{a_1}_{b_3} \\ & \quad + Iu_{a_1c}u_{a_3c}(u_{b_1c}u_{b_2c})^*\delta^{a_3}_{b_3} + Ju_{a_1c}u_{a_3c}(u_{b_1c}u_{b_3c})^*\delta^{a_2}_{b_2} \\ & \quad + Ku_{a_2c}u_{a_3c}(u_{b_1c}u_{b_3c})^*\delta^{a_1}_{b_2} + Lu_{a_1c}u_{a_2c}(u_{b_1c}u_{b_3c})^*\delta^{a_3}_{b_2} \\ & \quad + Mu_{a_2c}u_{a_3c}(u_{b_2c}u_{b_3c})^*\delta^{a_1}_{b_1} + Nu_{a_1c}u_{a_3c}(u_{b_2c}u_{b_3c})^*\delta^{a_2}_{b_1} \\ & \quad + Pu_{a_1c}u_{a_2c}(u_{b_2c}u_{b_3c})^*\delta^{a_3}_{b_1}] + \sum_c Ru_{a_1c}u_{a_2c}u_{a_3c}(u_{b_1c}u_{b_2c}u_{b_3c})^*. \end{aligned} \tag{A.15}$$

Suppose we choose the indices in (A.15) as

$$(a_1a_2a_3) = (112) \qquad (b_1b_2b_3) = (112). \tag{A.16}$$

Then (A.15) becomes

$$\begin{aligned} G = \sum_c [& G|u_{1c}|^4 + Hu_{1c}u_{2c}(u^2_{1c})^*\delta^1_2 + Iu_{1c}u_{2c}(u^2_{1c})^*\delta^1_2 + Ju_{1c}u_{2c}(u_{1c}u_{2c})^*\delta^1_1 \\ & + Ku_{1c}u_{2c}(u^2_{1c})^*\delta^1_1 + Lu^2_{1c}(u_{1c}u_{2c})^*\delta^2_1 + Mu_{1c}u_{2c}(u_{1c}u_{2c})^*\delta^1_1 \\ & + Nu_{1c}u_{2c}(u^2_{1c})^*\delta^1_1 + Pu^2_{1c}(u_{1c}u^*_{2c})\delta^2_1 + Ru^2_{1c}u_{2c}(u^2_{1c}u_{2c})^*]. \end{aligned} \tag{A.17}$$

On the right-hand side of (A.17) we observe the following property of the various terms beyond the first one: apart from the Kronecker deltas which make various of them vanish, they all contain u matrix elements associated with the rows 1 and 2: i.e. u_{1c} *and* u_{2c}.

Therefore, if we choose $\mathbf{u}$ as

$$\mathbf{u}=\left[\begin{array}{ccc|ccc} 1/\sqrt{2} & 0 & 1/\sqrt{2} & & & \\ 0 & 1 & 0 & & 0 & \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & & & \\ \hline & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right] \tag{A.18}$$

every term on the right-hand side of (A.17) vanishes, except the first one; we thus get $G=0$.

A similar analysis can be carried out for the choice

$$(a_1a_2a_3)=(211) \qquad (b_1b_2b_3)=(112) \tag{A.19}$$

to show $H=0$. Similarly, one finds that all the coefficients in (A.15) up to P vanish. Finally, we choose

$$(a_1a_2a_3)=(111) \qquad (b_1b_2b_3)=(111). \tag{A.20}$$

Since $G=\ldots=P=0$, (A.15) gives

$$R=R\sum_c |u_{1c}|^6 \tag{A.21}$$

$\forall\mathbf{u}$, so that $R=0$.

In conclusion, the most general q satisfying (A.1) for $m=3$ contains the first six terms of (A.14).

The generalisation to arbitrary m is now clear. Equation (A.14) becomes

$$\begin{aligned} q^{a_1a_2a_3\ldots a_m}_{b_1b_2b_3\ldots b_m} = {} & (A\Delta^{a_1a_2a_3\ldots a_m}_{b_1b_2b_3\ldots b_m} + B\Delta^{a_{1'}a_{2'}a_{3'}\ldots a_{m'}}_{b_1b_2b_3\ldots b_m}+\ldots) \\ & + \delta_{b_1b_2}(G\Delta^{a_1a_2a_3\ldots a_m}_{b_1b_1b_3\ldots b_m} + H\Delta^{a_{1'}a_{2'}a_{3'}\ldots a_{m'}}_{b_1b_1b_3\ldots b_m}+\ldots) \\ & + \delta_{b_1b_3}(J\Delta^{a_1a_2a_3\ldots a_m}_{b_1b_2b_1\ldots b_m} + K\Delta^{a_{1'}a_{2'}a_{3'}\ldots a_{m'}}_{b_1b_2b_1\ldots b_m}+\ldots) \\ & +\ldots+\delta_{b_1b_2}\delta_{b_2b_3}(R\Delta^{a_1a_2a_3a_4\ldots a_m}_{b_1b_1b_1b_4\ldots b_m} + S\Delta^{a_{1'}a_{2'}a_{3'}a_{4'}\ldots a_{m'}}_{b_1b_1b_1b_4\ldots b_m}+\ldots)+\ldots . \end{aligned} \tag{A.22}$$

In the above equation, the lines beyond the first one correspond to two, three, etc coinciding indices.

We now apply the transformation (A.1) to (A.22), whose first line, as usual, satisfies (A.1) identically. As an example, the term that multiplies H in (A.22) transforms as

$$\begin{aligned} & \sum_{\substack{\bar a_1\ldots\bar a_m \\ \bar b_1\ldots\bar b_m}} (u_{a_1\bar a_1}u_{a_2\bar a_2}u_{a_3\bar a_3}\ldots u_{a_m\bar a_m})^*(u_{b_1\bar b_1}u_{b_2\bar b_1}u_{b_3\bar b_3}\ldots u_{b_m\bar b_m})\Delta^{\bar a_{1'}\bar a_{2'}\bar a_{3'}\ldots\bar a_{m'}}_{\bar b_1\bar b_1\bar b_3\ldots\bar b_m} \\ & \qquad = \sum_c (u_{a_{1'}c}u_{a_{2'}c})^*(u_{b_1c}u_{b_2c})\Delta^{a_{3'}\ldots a_{m'}}_{b_3\ldots b_m}. \end{aligned} \tag{A.23}$$

We thus have

$$\begin{aligned} & \delta_{b_1b_2}(G\Delta^{a_1a_2a_3\ldots a_m}_{b_1b_1b_3\ldots b_m} + H\Delta^{a_{1'}a_{2'}a_{3'}\ldots a_{m'}}_{b_1b_1b_3\ldots b_m}+\ldots) \\ & \qquad + \delta_{b_1b_3}(J\Delta^{a_1a_2a_3\ldots a_m}_{b_1b_2b_1\ldots b_m} + K\Delta^{a_{1'}a_{2'}a_{3'}\ldots a_{m'}}_{b_1b_2b_1\ldots b_m}+\ldots) \\ & \qquad +\ldots+\delta_{b_1b_2}\delta_{b_2b_3}(R\Delta^{a_1a_2a_3a_4\ldots a_m}_{b_1b_1b_1b_4\ldots b_m} + S\Delta^{a_{1'}a_{2'}a_{3'}a_{4'}\ldots a_{m'}}_{b_1b_1b_1b_4\ldots b_m}+\ldots)+\ldots \\ & \quad = \sum_c [G(u_{a_1c}u_{a_2c})^*(u_{b_1c}u_{b_2c})\Delta^{a_3\ldots a_m}_{b_3\ldots b_m} + H(u_{a_{1'}c}u_{a_{2'}c})^*(u_{b_1c}u_{b_2c})\Delta^{a_{3'}\ldots a_{m'}}_{b_3\ldots b_m} \\ & \qquad +\ldots+J(u_{a_1c}u_{a_3c})^*(u_{b_1c}u_{b_3c})(\Delta^{a_2a_4\ldots a_m}_{b_2b_4\ldots b_m} + K(u_{a_{1'}c}u_{a_{3'}c})^*(u_{b_1c}u_{b_3c})\Delta^{a_{2'}a_{4'}\ldots a_{m'}}_{b_2b_4\ldots b_m} \\ & \qquad +\ldots+R(u_{a_1c}u_{a_2c}u_{a_3c})^*(u_{b_1c}u_{b_2c}u_{b_3c})\Delta^{a_4\ldots a_m}_{b_4\ldots b_m} \\ & \qquad + S(u_{a_{1'}c}u_{a_{2'}c}u_{a_{3'}c})^*(u_{b_1c}u_{b_2c}u_{b_3c})\Delta^{a_{4'}\ldots a_{m'}}_{b_4\ldots b_m}+\ldots] \end{aligned} \tag{A.24}$$

which generalises (A.15).

If in (A.22) we make the choice of indices

$$(a_1 a_2 a_3 \ldots a_m) = (112 \ldots 2) \qquad (b_1 b_2 b_3 \ldots b_m) = (112 \ldots 2) \tag{A.25}$$

the term with G is singled out on the left-hand side, just as in (A.17). On the right-hand side, the first term is $G\Sigma_c |u_{1c}|^4$ and, just as in (A.17), all the other terms contain u matrix elements associated with rows 1 *and* 2 and thus give a vanishing contribution if $\mathbf{u}$ is chosen as in (A.18). We thus conclude that $G = 0$.

Appropriate choices of the indices then show, just as for $m = 3$, that the remaining coefficients vanish too. As a result, the first line in (A.22), i.e.

$$q^{a_1 \ldots a_m}_{b_1 \ldots b_m} = A\Delta^{a_1 \ldots a_m}_{b_1 \ldots b_m} + B\Delta^{a_1' \ldots a_m'}_{b_1 \ldots b_m} + \ldots \tag{A.26}$$

is the most general q satisfying (A.1).

It is interesting to view (A.26) as a generalisation of Schur's lemma mentioned for $m = 1$. For general m, (A.1) can be written as

$$\mathbf{q}(\mathbf{u} \times \ldots \times \mathbf{u}) = (\mathbf{u} \times \ldots \times \mathbf{u})\mathbf{q} \tag{A.27}$$

using the matrix notation employed in (A.6) and (A.7). In (A.27) each $\mathbf{u}$ acts on a pair of indices $a_i b_i$ of $\mathbf{q}$. Since the direct product $\mathbf{u} \times \ldots \times \mathbf{u}$ is not irreducible, Schur's lemma does not apply, as it did for $m = 1$; what we have proved instead is that $\mathbf{q}$ must be of the form (A.26).

We recall that we want to construct the object Q that satisfies condition I of (2.3a) *and* (2.3b). If we start by requiring (2.3a), Q must have the form (A.26), the coefficients depending now on the greek indices. That is

$$Q^{a_1 \alpha_1, \ldots, a_m \alpha_m}_{b_1 \beta_1, \ldots, b_m \beta_m} = \Delta^{a_1 \ldots a_m}_{b_1 \ldots b_m} A^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_m} + \Delta^{a_1' \ldots a_m'}_{b_1 \ldots b_m} B^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_m} + \ldots . \tag{A.28}$$

If we now impose (2.3b), we see that $A^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_m}$, $B^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_m}$ must have the structure (A.26) with greek indices and constant coefficients. This proves the statement made in the first paragraph of this appendix.

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